

Note on a Nonlinear Differential Equation

Nikos Bagis
nikosbagis@hotmail.gr

Abstract

We give evaluations in closed form of certain non linear differential equations

1 The equation $y'''(x) + y'(x) = P(y(x))$

We consider the folowing non linear differential equation (NLDE)

$$y'''(x) + y'(x) = \frac{1}{2}P(y(x)) \quad (1)$$

where $P(x)$ is polynomial or a function of the form

$$P(x) = \left(\sum_{n=0}^N a_n x^n \right) \sqrt{\sum_{n=0}^M b_n x^n} \quad (2)$$

The non linear equation (1) is equivalent to

$$h''(x) = \frac{P(x)}{\sqrt{h(x) - x^2}}. \quad (3)$$

and if the above equation has algebraic solution

$$h(x) = x^2 + U^2(x),$$

then

$$y_i(x) = \frac{1}{2} \int \frac{h''(x)}{P(x)} dx$$

where y_i denotes the inverse of $y(x)$ i.e $y_i(x) = y^{(-1)}(x)$.

Proof.

We define by $F(x)$ the function such that

$$y''(x) = F(y(x)) - y(x) \quad (4)$$

then (4) has solution

$$\int_{C_3}^x \frac{1}{\sqrt{C_1 + 2 \int_{C_2}^t (F(v) - v) dv}} dt = y_i(x) \quad (5)$$

Also relating (1) with (5) we get

$$\int_C^x \frac{F'(t)}{P(t)} dt = yi(x) \quad (6)$$

Again relating (4) with (5) we have

$$\frac{F'(x)}{P(x)} = \frac{1}{\sqrt{C_1 + 2 \int_{C_2}^x (F(v) - v) dv}} \quad (7)$$

Then if

$$h(x) - x^2 = U^2(x) = C_1 + 2 \int_{C_2}^x (F(t) - t) dt \quad (8)$$

we get

$$h''(x) = \frac{P(x)}{\sqrt{h(x) - x^2}}$$

The idea behind the above formulation is that last equation admits polynomial solutions if $P(x)$ is of type (2).

Let

$$P(x) = \left(\sum_{n=0}^N a_n x^n \right) \sqrt{\sum_{n=0}^M b_n x^n}, \quad (9)$$

then b_n are depent from a_n with very simple way.

Set where $h(x)$ in (3) the polynomial $x^2 + U(x)^2$, with

$$U(x) = \sqrt{\sum_{n=0}^N b_n x^n} \quad (10)$$

and evaluate the a_n such

$$h''(x) - \sum_{n=0}^N a_n x^n = 0 \quad (11)$$

or equivalently

$$2 + 2U'^2(x) + 2U(x)U''(x) = \sum_{n=0}^N a_n x^n \quad (12)$$

since

$$h(x) = x^2 + U^2(x). \quad (13)$$

This can be done as in the following

Example 1. Set

$P(x) = (g + fx + ex^2)(g_1 + f_1x + e_1x^2)$ and $U(x) = g + fx + ex^2$, then if $h(x) = x^2 + U^2(x)$, $e_1 = 12e^2$, $f_1 = 12ef$, $g_1 = 2 + 2f^2 + 4eg$, we get

$$h(x) = x^2 + (g + fx + ex^2)^2.$$

Hence if we have to solve the equation

$$y'''(x) + y'(x) = g + 2eg^2 + (e + 8e^2g)y^2(x) + 6e^3y^4(x)$$

then $P(x) = g + f^2g + 2eg^2 + (f + f^3 + 8efg)x + (e + 7ef^2 + 8e^2g)x^2 + 12e^2fx^3 + 6e^3x^4$, $h(x) = x^2 + (g + fx + ex^2)^2$ and

$$y_i(x) = \int \frac{h''(x)}{P(x)} dx = \frac{2 \arctan\left(\frac{2ex+f}{\sqrt{4eg-f^2}}\right)}{\sqrt{4eg-f^2}}$$

Finally inverting we get

$$y(x) = \frac{\sqrt{4eg-f^2} \tan\left(\frac{x}{2}\sqrt{4eg-f^2}\right) - f}{2e}$$

Example 2. The equation

$$y'''(x) + y'(x) = dy^3(x) + 15d^3y^7(x)$$

have $P(x) = 2dx^3 + 15d^3x^7$ and $h(x) = x^2 + d^2x^6$ and hence

$$y(x) = \frac{i}{\sqrt{2d \cdot x}}.$$

Also another equation is with

$$P(x) = 15d^3x^7 + 21d^2gx^4 + dx^3 + 6dg^2x + g$$

then $h(x) = x^2 + (g + dx^3)^2$ and the solution is such that

$$\begin{aligned} y_i(x) = & \\ = & \frac{1}{3\sqrt[3]{dg^{2/3}}} [45d^{4/3}g^{2/3}x^2 - (9d^{2/3}g^{4/3} + 1) \log(d^{2/3}x^2 - \sqrt[3]{d}\sqrt[3]{g}x + g^{2/3}) + \\ & + 2(9d^{2/3}g^{4/3} + 1) \log(\sqrt[3]{d}x + \sqrt[3]{g}) + 2\sqrt{3}(9d^{2/3}g^{4/3} - 1) \arctan\left(\frac{1 - \frac{2\sqrt[3]{dx}}{\sqrt[3]{g}}}{\sqrt{3}}\right)] \end{aligned}$$

Example 3. Assume the NLDE with

$$\begin{aligned} P(x) = & \frac{1}{2\sqrt{210}} (a_1x^6 + b_1x^5 + c_1x^4 + d_1x^3 + e_1x^2 + f_1x + g_1) \times \\ & \times [15a_1x^8 + 20b_1x^7 + 28c_1x^6 + 42d_1x^5 + 70e_1x^4 + 140f_1x^3 + 420g_1x^2 - \\ & - 840x^2 + 840C_2x + 1680C_1]^{1/2} \end{aligned}$$

then

$$C + x = 2\sqrt{210} \int_0^{y(x)} [15a_1t^8 + 20b_1t^7 + 28c_1t^6 + 42d_1t^5 + 70e_1t^4 + 140f_1t^3 + (420g_1 - 840)t^2 + 840C_2t + 1680C_1]^{-1/2} dt$$

An application of the above example is taking

$$P(x) = \frac{(f_1x + g_1)\sqrt{1680C_1 + 840C_2x + 140f_1x^3 + 420g_1x^2 - 840x^2}}{2\sqrt{210}}$$

then

$$h(x) = x^2 + \frac{1}{840} (1680C_1 + 840C_2x + (420g_1 - 840)x^2 + 140f_1x^3)$$

and $y(x)$ is given from

$$x + C = \int_0^{y(x)} \frac{dt}{\sqrt{\frac{f_1}{6}t^3 + \frac{(g_1-2)}{2}t^2 + C_2t + 2C_1}}$$

The above integral can be evaluated using the incomplete elliptic integral of the first kind $F[x, m]$, i.e

$$F[x, m] = \int_0^x \frac{dt}{\sqrt{1 - m \sin^2(t)}}. \quad (14)$$

The reader can see [1].

2 References

[1]: J.V. Armitage W.F. Eberlein. 'Elliptic Functions'. Cambridge University Press. (2006)